Unit Vectors (covered in § 12.2)

vector:

$$\vec{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$$

length of $\vec{v}$ $\quad \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

(Pythagorean Thm)

Unit Vectors are
Vectors with unit length (length = 1)

$\vec{v} \in \mathbb{R}^n$ can be normalized to
a unit vector $\dfrac{\vec{v}}{\|\vec{v}\|}$. $\left(\left\| \dfrac{\vec{v}}{\|\vec{v}\|} \right\| = \dfrac{\|\vec{v}\|}{\|\vec{v}\|} = 1 \right)$

(unit vector can be understand as direction,
normalized vector is the direction of orginal vector)

Standard Unit Vectors:

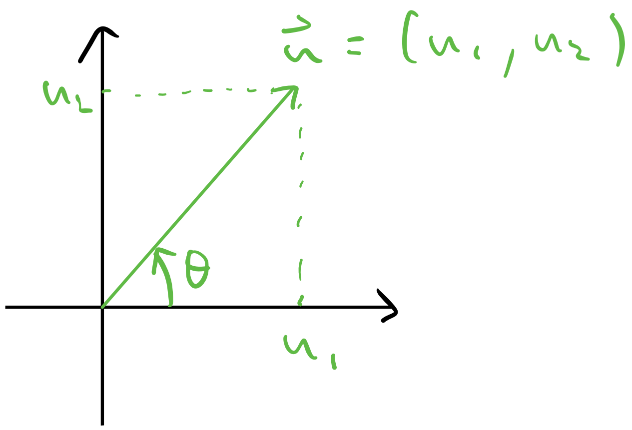
$\mathbb{R}^2$: $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$.

$\mathbb{R}^3$: $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$,
$\vec{k} = (0, 0, 1)$.

Every Vector $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$
can be written as $v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$,
same for $\mathbb{R}^2$

§12.2

56. Unit vectors in the plane Show that a unit vector in the plane can be expressed as $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, obtained by rotating $\mathbf{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives *every* unit vector in the plane.



$\vec{u}$: unit vector

Then $u_1 = \|\vec{u}\|\cos\theta$

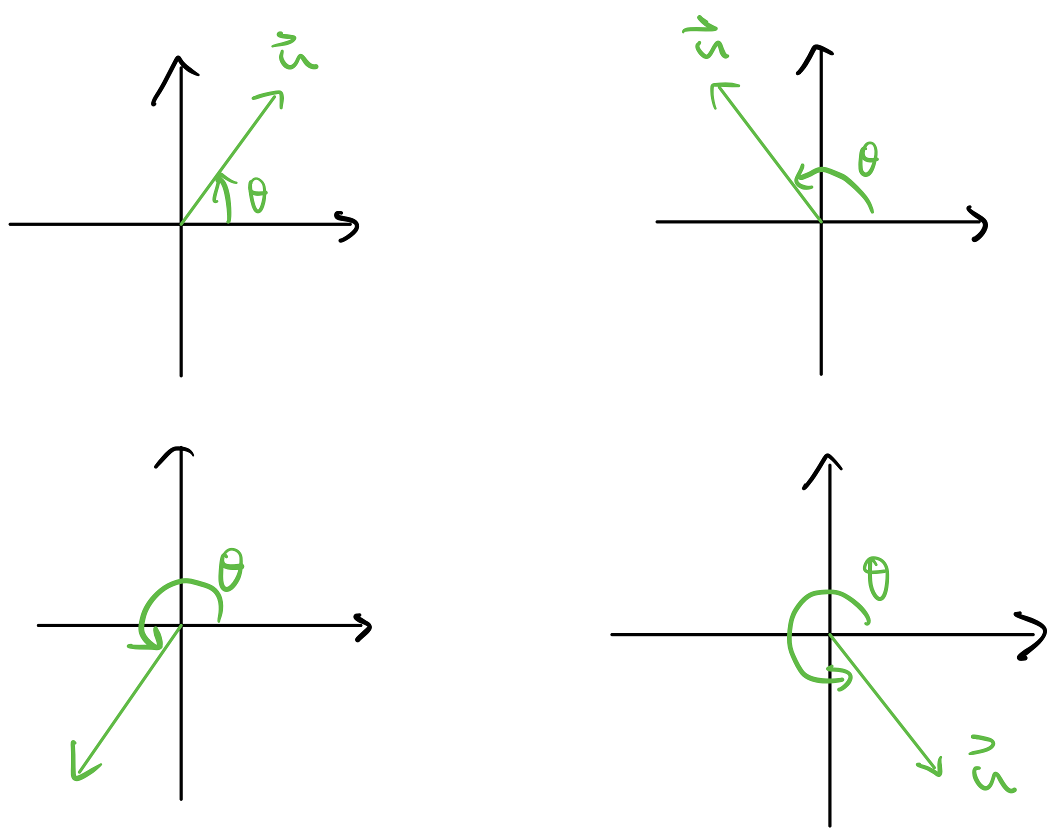
$= \cos\theta$

$u_2 = \|\vec{u}\|\sin\theta$

$= \sin\theta$

$\therefore \vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$.

Counter Clockwise Direction:

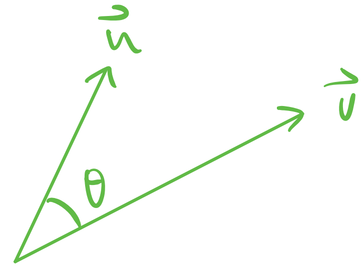


Dot Product, Projection (Covered in §12.3)

Dot Product : $\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

Remark : $\vec{u} \cdot \vec{u} = \| \vec{u} \|^2$

The angle between $\vec{u}$ and $\vec{v}$:

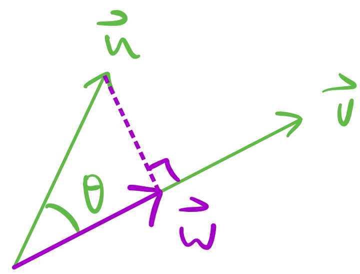


$$\theta = \cos^{-1} \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \right)$$

(pf. : using cosine formula :
$\| \vec{v} - \vec{u} \|^2 = \|\vec{u}\|^2 + \| \vec{v} \|^2 - 2 \|\vec{u}\| \, \|\vec{v}\| \cos\theta$)

Remark : $\vec{u} \cdot \vec{v} = 0 \iff \vec{u}, \vec{v}$ are orthogonal

Projection of $\vec{u}$ onto $\vec{v}$:



a vector $\vec{w}$ has direction same as $\vec{v}$, and length $\|\vec{u}\| \cos\theta$

$$\vec{w} = \underbrace{\|\vec{u}\| \cos\theta}_{\text{length}} \cdot \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{direction}}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

We can also write $\vec{w} = \mathrm{proj}_{\vec{v}} \vec{u}$

§12.3

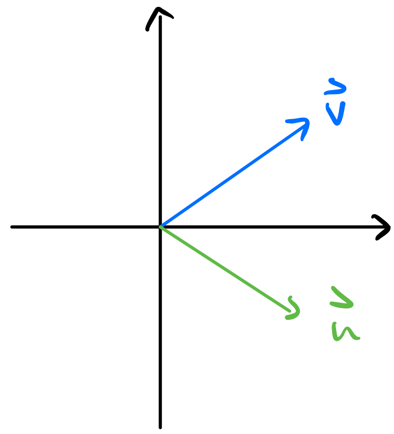
8. $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

a. $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$

b. the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$

c. the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

d. the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.



(a). $\vec{v} \cdot \vec{u} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \left(-\frac{1}{\sqrt{3}}\right)$

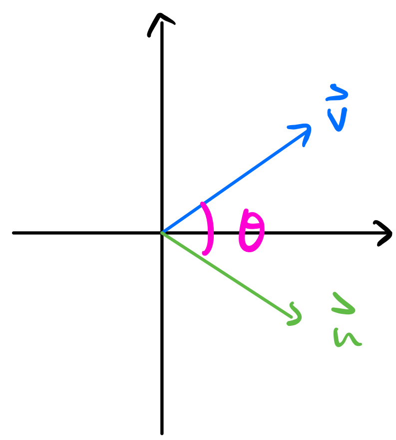
$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$

$\|\vec{v}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{5}{6}}$

$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{5}{6}}.$

(b) $\cos\theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \cdot \|\vec{u}\|}$

$= \frac{1/6}{\sqrt{5/6}\,\sqrt{5/6}} = \frac{1}{5}$

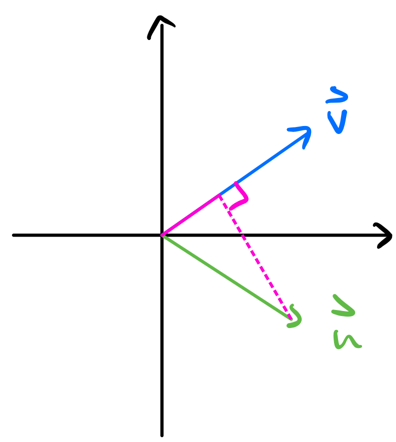


(c). scalar component

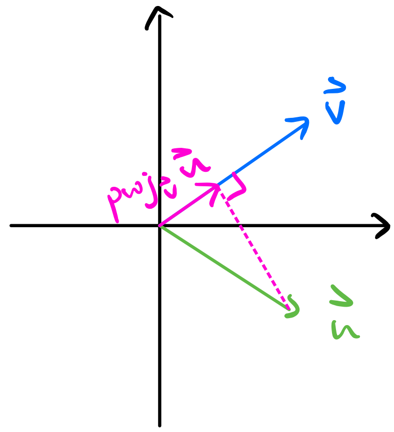
$=$ length of projection

$= \|\vec{u}\| \cos\theta$

$= \sqrt{\frac{5}{6}} \cdot \frac{1}{5} = \frac{1}{\sqrt{30}}$



(d). $\text{Proj}_{\vec{v}} \vec{u}$



$$= \|\vec{u}\| \cos\theta \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{1}{\sqrt{30}} \cdot \frac{\sqrt{6}}{\sqrt{5}} \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)$$

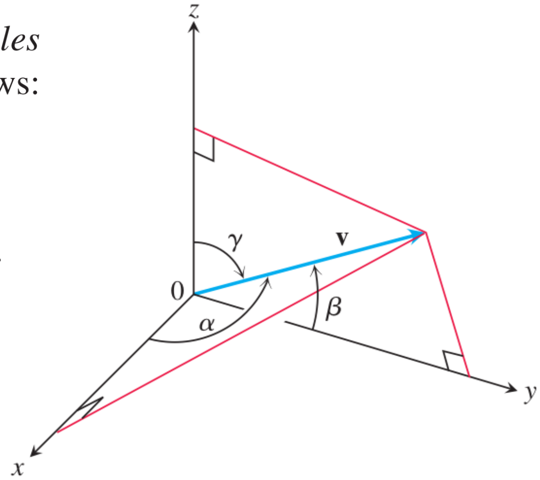
$$= \left(\frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{3}} \right)$$

15. Direction angles and direction cosines The *direction angles* α, β, and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:

α is the angle between $\mathbf{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

β is the angle between $\mathbf{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

γ is the angle between $\mathbf{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$.



a. Show that

$$\cos\alpha = \frac{a}{|\mathbf{v}|}, \qquad \cos\beta = \frac{b}{|\mathbf{v}|}, \qquad \cos\gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the *direction cosines* of $\mathbf{v}$.

b. Unit vectors are built from direction cosines Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a, b, and c are the direction cosines of $\mathbf{v}$.

(a). $\cos\alpha = \dfrac{\vec{i}\cdot\vec{v}}{\|\vec{i}\|\,\|\vec{v}\|}$

$$= \frac{(\vec{i}\cdot a\vec{i}) + (\vec{i}\cdot b\vec{j}) + (\vec{i}\cdot c\vec{k})}{\|\vec{v}\|}$$

$$= \frac{a + 0 + 0}{\|\vec{v}\|} = \frac{a}{\|\vec{v}\|}$$

Similar for $\cos\beta$ and $\cos\gamma$.

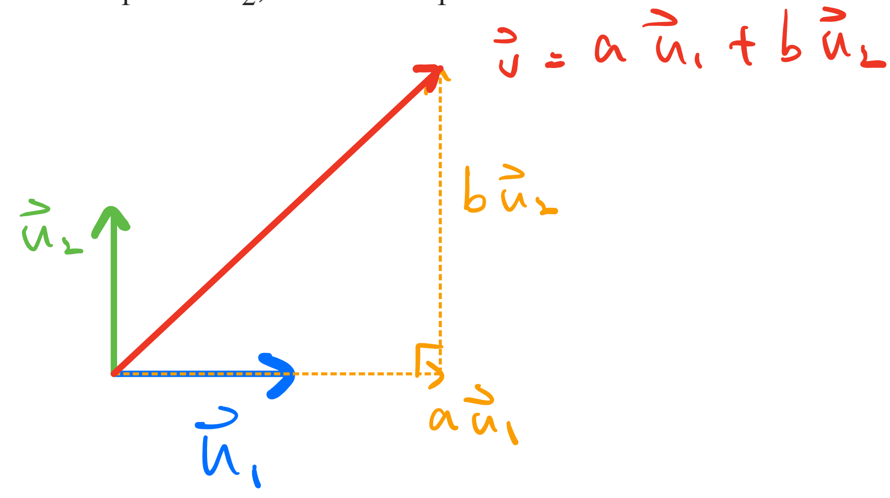
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{a^2 + b^2 + c^2}{\|\vec{v}\|^2} = \underline{1}.$$

(b). If $\vec{u}$ is unit, then $\cos\alpha = \dfrac{a}{\|\vec{v}\|} = a$

$\cos\beta = \dfrac{b}{\|\vec{v}\|} = b$, $\cos\gamma = \dfrac{c}{\|\vec{v}\|} = c$

27. Orthogonal unit vectors If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors and $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2$, find $\mathbf{v} \cdot \mathbf{u}_1$.



$a\vec{u}_1$ is projection of $\vec{v}$ onto $\vec{u}_1$:

$$a\vec{u}_1 = \frac{\vec{v} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\,\vec{u}_1$$

$$= (\vec{v} \cdot \vec{u}_1)\,\vec{u}_1$$

Or

Recall: $\vec{u}_1, \vec{u}_2$ are orthogonal if $\vec{u}_1 \cdot \vec{u}_2 = 0$.

$$\vec{v} \cdot \vec{u}_1$$

$$= (a\vec{u}_1 + b\vec{u}_2) \cdot \vec{u}_1$$

$$= a\,\vec{u}_1 \cdot \vec{u}_1 + b\,\vec{u}_2 \cdot \vec{u}_1$$

$$= a$$

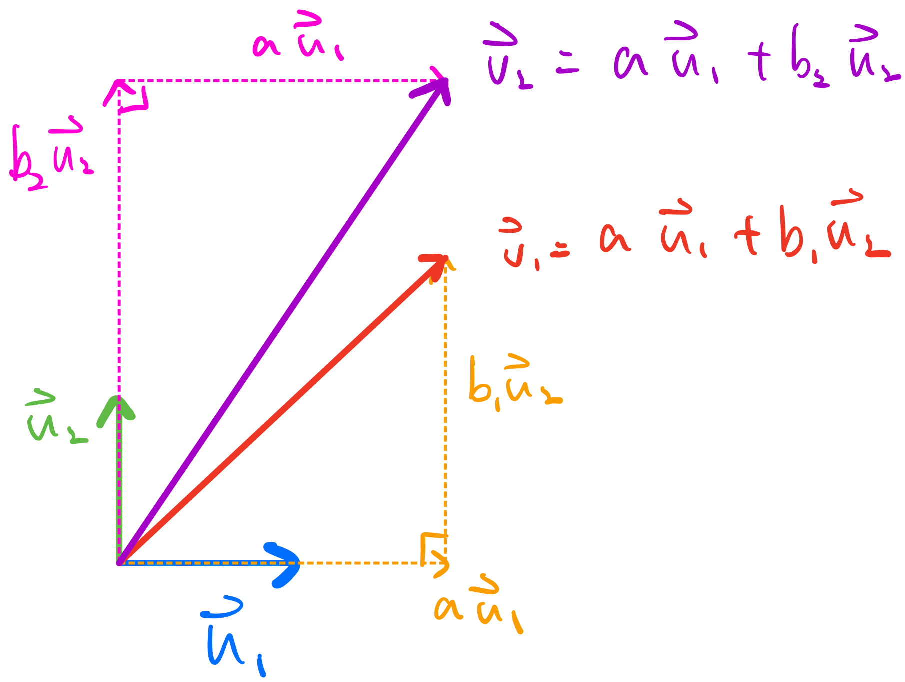
28. Cancellation in dot products In real-number multiplication, if $uv_1 = uv_2$ and $u \neq 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product? That is, if $\mathbf{u} \cdot \mathbf{v}_1 = \mathbf{u} \cdot \mathbf{v}_2$ and $\mathbf{u} \neq \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

Recall the meaning of dot product:

length of projection onto a (unit) vector.

Under the same setting with previous question

($\vec{u}_1, \vec{u}_2$ are orthogonal unit vectors),



Both $\vec{v}_1$ and $\vec{v}_2$ have the same projection onto $\vec{u}_1$

By previous question,

$\vec{u}_1 \cdot \vec{v}_1 = \vec{u}_1 \cdot \vec{v}_2$ but $\vec{v}_1 \neq \vec{v}_2$.

Or Counter Example:

Let $u = (1, 0)$, $v_1 = (1, 1)$

$v_2 = (1, 2)$

$\vec{u} \cdot \vec{v}_1 = \underline{1} = \vec{u} \cdot \vec{v}_2$.

but $\vec{v}_1 \neq \vec{v}_2$.